Descriptive complexity of self-reducible counting problems with an easy decision version*

Antonis Achilleos[1] and Aggeliki Chalki[2]

[1] Reykjavik University, Reykjavik, Iceland
antonios@ru.is
[2] National Technical University of Athens, Athens, Greece
Reykjavik University, Reykjavik, Iceland
angelikic@ru.is

Introduction. The class $\#\mathsf{P}$ was defined in Valiant's seminal paper [1] to capture the complexity of computing the permanent of a square matrix and it is the class of counting versions of NP problems. A function $f : \Sigma^* \to \mathbb{N}$ belongs to $\#\mathsf{P}$ if there is a non-deterministic polynomial-time Turing machine (NPTM) M such that for every $x \in \Sigma^*$, $f(x) = \#(\text{accepting paths of } M \text{ on } x)$. And since accepting paths correspond to solutions of computational problems, a function in $\#\mathsf{P}$ is the problem of counting the number of solutions of the corresponding NP problem. For example, $\#$SAT is a function that maps an input formula to the number of its satisfying assignments.

Since very few counting problems can be exactly computed in polynomial time—an example is counting spanning trees in a graph, which can be reduced to computing the determinant of a matrix—the interest of the community has turned to the complexity of approximating them. To this end, the class $\#\mathsf{PE}$ [2] is of great significance. $\#\mathsf{PE}$ contains $\#\mathsf{P}$ functions the decision versions of which are in P—i.e. for any $f \in \#\mathsf{PE}$, we can decide whether a given x is a zero value of f in polynomial time. To the best of our knowledge, almost all approximable problems have a decision version in P.

We focus on a subclass of $\#\mathsf{PE}$, namely TotP [3], that is the class of functions that count the total number of paths of NPTMs. Notably, TotP contains all self-reducible $\#\mathsf{PE}$ functions [4], and it is *robust* [5], in the sense that it has natural complete problems [6] and it is closed under addition, multiplication and subtraction by one. The connection of the class of approximable counting problems to TotP has been emphasized in [5], and the relationship of these two classes has been examined in [7].

In this work we address the open question of [5] about the descriptive complexity of TotP. We provide a logical characterization of TotP based on the framework of Quantitative logics introduced in [5]. Descriptive complexity is an interesting area of complexity since it links syntactic restrictions for specifications to general algorithmic guarantees. Fagin's fundamental theorem [8] gives a correspondence between the class NP and the class of problems that can be expressed in existential second-order logic, denoted by $\exists\mathsf{SO}$, over finite structures. So, existential quantification over second-order variables suffices in order to express the existence of a solution to an NP problem. It is a natural consequence that counting the number of second-order variables for which a formula is satisfied in some structure, results in a logical characterization of $\#\mathsf{P}$. Recently, Arenas et al. [5] incorporated counting into the syntax of the logic by allowing the use of the quantitative quantifiers Σ and Π—an addition and a multiplication quantifier, respectively—in addition to the boolean quantifiers $\exists$ and $\forall$. In this context, $\Sigma\mathsf{QSO}(\mathsf{FO}) = \#\mathsf{P}$

*This work has been funded by the Basic Research Program PEVE 2020 of the National Technical University of Athens, and by the projects "Open Problems in the Equational Logic of Processes (OPEL)" (grant no 196050) and "Mode(l)s of Verification and Monitorability" (MoVeMent) (grant no 217987) of the Icelandic Research Fund.

over finite ordered structures. The part ΣQSO reflects the fact that Σ quantitative quantifier is used over second-order variables (whereas Π quantifier is not allowed) and (FO) indicates that boolean quantifiers are used only over first-order variables. An example is given in Table 1. Arenas et al. also added recursion over functions at the quantitative level to capture the classes FP—the class of polynomial-time computable functions—and #L [9].

$\exists$SO = NP over finite structures	G contains a clique of any size iff $\mathcal{G} \models \exists X \forall x \forall y \big(X(x) \wedge X(y) \wedge x \neq y\big) \rightarrow E(x, y)$
ΣQSO(FO) = #P over finite ordered structures	$\#(\text{cliques of } G) =$ $[[\Sigma X.\forall x \forall y \big(X(x) \wedge X(y) \wedge x \neq y\big) \rightarrow E(x, y)]](\mathcal{G})$

Table 1: The decision and counting versions of the Clique problem expressed in the logics introduced by Fagin and Arenas et al., respectively. The input structure $\mathcal{G}$ is over vocabulary $\langle E^2 \rangle$ with one relational symbol E of arity 2 representing the edge relation.

Descriptive complexity of the class TotP. The idea we are using to capture TotP is the following: 1. Start with a logic that expresses the computation of an NPTM, 2. If a branching is found during the computation, add 1, and then continue recursively, and 3. Stop after a polynomial number of steps. To achieve the recursion needed here, we generalize the notion of recursion defined in [5]; Instead of recursion over functions the arguments of which are first-order variables and the definition of a least-fixed point, we introduce recursion over functions that take second-order variables as arguments and we define a fixed-point of polynomial depth which is called *polynomially-bounded fixed point* (or *p-bounded fixed point*), denoted by pbfp.

Contribution. The logical characterization of TotP is related to the following major open question *"Can we provide a logical characterization of counting problems that are approximable?"*. Saluja et al. proved that given a formula which expresses a counting problem, it is undecidable whether the problem is approximable (under the widely believed assumption that RP $\neq$ NP) [10]. The study of the descriptive complexity of counting problems that are eligible to have efficient approximation algorithms—in specific, counting problems whose decision version is easy—has been a more fruitful approach [10, 5, 7]. The relationship of the classes defined in this context to the class of approximable counting problems has also been determined. The logical characterization of TotP, namely the class that contains self-reducible problems with an easy decision problem, still remains open. This work settles this open question by providing a logic that captures TotP over finite ordered structures.

References

[1] L. Valiant, "The complexity of computing the permanent," *Theoretical Computer Science*, vol. 8, no. 2, pp. 189–201, 1979. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0304397579900446

[2] A. Pagourtzis, "On the complexity of hard counting problems with easy decision version," in *Proceedings of the 3rd Panhellenic Logic Symposium, PLS 2001, Anogia, Greece, July 17-21, 2001*, 2001, pp. 21–29.

[3] A. Kiayias, A. Pagourtzis, K. Sharma, and S. Zachos, "Acceptor-definable counting classes," in *Proceedings of the 8th Panhellenic Conference on Informatics, PCI 2001*, ser. Lecture

Notes in Computer Science, vol. 2563. Springer, 2001, pp. 453–463. [Online]. Available: https://doi.org/10.1007/3-540-38076-0_29

[4] A. Pagourtzis and S. Zachos, "The complexity of counting functions with easy decision version," in *Proceedings of the 31st International Conference on Mathematical Foundations of Computer Science, MFCS 2006*. Springer, 2006, pp. 741–752. [Online]. Available: https://doi.org/10.1007/11821069_64

[5] M. Arenas, M. Muñoz, and C. Riveros, "Descriptive complexity for counting complexity classes," *Logical Methods in Computer Science*, vol. 16, no. 1, 2020. [Online]. Available: https://doi.org/10.23638/LMCS-16(1:9)2020

[6] A. Antonopoulos, E. Bakali, A. Chalki, A. Pagourtzis, P. Pantavos, and S. Zachos, "Completeness, approximability and exponential time results for counting problems with easy decision version," *Theoretical Computer Science*, vol. 915, pp. 55–73, 2022. [Online]. Available: https://doi.org/10.1016/j.tcs.2022.02.030

[7] E. Bakali, A. Chalki, and A. Pagourtzis, "Characterizations and approximability of hard counting classes below #P," in *Proceedings of the 16th International Conference on Theory and Applications of Models of Computation, TAMC 2020*, ser. Lecture Notes in Computer Science, vol. 12337. Springer, 2020, pp. 251–262. [Online]. Available: https://doi.org/10.1007/978-3-030-59267-7_22

[8] R. Fagin, "Generalized first-order spectra, and polynomial. time recognizable sets," *SIAM-AMS Proceedings*, vol. 7, pp. 43–73, 1974. [Online]. Available: http://www.researchgate.net/publication/242608657_Generalized_first-order_spectra_and_polynomial._time_recognizable_sets

[9] C. Àlvarez and B. Jenner, "A very hard log-space counting class," *Theor. Comput. Sci.*, vol. 107, no. 1, pp. 3–30, 1993. [Online]. Available: https://doi.org/10.1016/0304-3975(93)90252-O

[10] S. Saluja, K. Subrahmanyam, and M. Thakur, "Descriptive complexity of #P functions," *Journal of Computer and System Sciences*, vol. 50, no. 3, pp. 493–505, 1995. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0022000085710392